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Analytical schemes for a new class of fractional differential equations

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Abstract

Fractional differential equations (FDEs) considered so far contain mostly left (or forward) fractional derivatives. In this paper, we present analytical solutions for a class of FDEs which contain both the left and the right (or the forward and the backward) fractional derivatives. The methods presented use properties of fractional integral operators (which, in many cases, lead to Volterra-type integral equations), an operational approach and a successive approximation method to obtain the solutions. The methods are demonstrated using some examples. The FDEs considered may come from fractional variational calculus (FVC) or from other physical principles. In the case of fractional variational problems (FVPs), the transversality conditions are used to identify appropriate boundary conditions and to solve the problems. It is hoped that this study will lead to further investigations in the field and more elegant solutions would be found.

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1. Introduction

In the last two decades, fractional derivatives (or fractional calculus) have played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, and notably control theory, robotics, image and signal processing [1–6]. Application of fractional derivatives (FDs) to modelling of many of these systems leads to fractional differential equations (FDEs) [2, 6]. Several analytical techniques have been developed to solve these equations which include integral transforms (such as Laplace, Mellin and Fourier), Greens function, operational methods, series solutions and others [1, 2, 6–11]. The FDEs considered in these books, monographs and special issues include only one type of FDs and mostly the left (or forward) FDs. For simplicity in the discussion to follow, we will call the FDEs that contain only one kind of FDs as Type1 FDEs. In many applications, the resulting

FDEs may contain both the left and the right (or the forward and the backward) FDs [12, 13]. We will call such FDEs as Type2 FDEs. To our knowledge, only a few papers have considered analytical solutions of Type2 FDEs [12, 13] that are very simple.

In this paper, we present analytical techniques to solve Type2 FDEs that are more general than those considered in the past [12, 13]. We utilize properties of fractional integrals (which, in many cases, lead to Volterra-type integral equations), operational method and successive approximation to develop the methods. Class2 FDEs considered here arise in fractional variational calculus (FVC) [12, 13]. Note that many integer-order differential equations come from physical principles that do not have variational formulations. Therefore, we believe that many future Type2 FDEs would also come from other sources. The methods are demonstrated using examples.

To solve differential equations (integer or Type1/Type2 fractional order), terminal or boundary conditions are necessary. When an integer-order differential equation is obtained using a variational formulation, the transversality conditions provide the natural boundary conditions. Such formulations for FVPs have recently been presented in [13]. It is demonstrated that the transversality conditions provide a way to identify suitable natural boundary conditions for Type2 FDEs [13].

It should be emphasized that the focus of this paper is only on analytical techniques for Type2 FDEs. However, some papers have recently presented numerical techniques for this class of problems (see, for example, [14, 15]). As pointed out above, FVPs lead to Type2 FDEs. Some direct numerical techniques to solve FVPs, which do not obtain FDEs, could be found in [16, 17]. Other fractional variational formulations (which also lead to Type2 FDEs) and their applications in various fields of science and engineering could be found, for example, in [12, 13] and the references cited therein.

To develop our methods, in the next section, we present the definitions of derivatives and integrals of fractional orders and their properties relevant to this work.

2. Fractional derivatives and their properties

Several definitions have been proposed for a fractional derivative. We will deal with the Riemann–Liouville and the Caputo fractional derivatives only. In this section, we present the definitions of these two derivatives and their properties pertinent to this research.

We begin with the left and the right Riemann–Liouville fractional integrals of order $\alpha > 0$ of a function $x(t)$ which are defined as [2]

$$ {}_0I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau)\,\mathrm{d}\tau, \qquad t, \alpha > 0 \tag{1} $$

and

$$ {}_tI_1^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (\tau-t)^{\alpha-1} x(\tau)\,\mathrm{d}\tau, \qquad t, \alpha > 0, \tag{2} $$

where $\Gamma(*)$ represents the gamma function. We have selected the limits as 0 and 1. However, other limits can also be selected. Using (1) and (2) the left and the right Riemann–Liouville derivatives ${}_0D_t^\alpha x(t)$ and ${}_tD_1^\alpha x(t)$ and the left and the right Caputo derivatives ${}_0^CD_t^\alpha x(t)$ and ${}_t^CD_1^\alpha x(t)$ of order $\alpha > 0$ are given as

$$ {}_0D_t^\alpha x(t) = D^n {}_0I_t^{n-\alpha} x(t), \qquad n-1 < \alpha < n, \tag{3} $$

$$ {}_tD_1^\alpha x(t) = (-D)^n {}_tI_1^{n-\alpha} x(t), \qquad n-1 < \alpha < n, \tag{4} $$

$$ {}_0^CD_t^\alpha x(t) = {}_0I_t^{n-\alpha} D^n x(t), \qquad n-1 < \alpha < n \tag{5} $$

and

$$ {}_t^C D_1^\alpha x(t) = {}_t I_1^{n-\alpha} (-D)^n x(t), \qquad n-1 < \alpha < n, \tag{6} $$

where $D = \mathrm{d}/\mathrm{d}t$ represents the ordinary derivative and n is an integer. These derivatives satisfy the following properties:

$$ {}_0 I_t^\alpha \, {}_0 D_t^\alpha x(t) = x(t) - \sum_{j=1}^{n} \frac{\left({}_0 D_t^{\alpha-j} x\right)(0)}{\Gamma(\alpha+1-j)} t^{\alpha-j}, \tag{7} $$

$$ {}_t I_1^\alpha \, {}_t D_1^\alpha x(t) = x(t) - \sum_{j=1}^{n} \frac{\left({}_t D_1^{\alpha-j} x\right)(1)}{\Gamma(\alpha+1-j)} (1-t)^{\alpha-j}, \tag{8} $$

$$ {}_0 I_t^\alpha \, {}_0^C D_t^\alpha x(t) = x(t) - \sum_{j=0}^{n-1} \frac{(D^j x)(0)}{\Gamma(j+1)} t^j, \tag{9} $$

$$ {}_t I_1^\alpha \, {}_t^C D_1^\alpha x(t) = x(t) - \sum_{j=0}^{n-1} \frac{((-D)^j x)(1)}{\Gamma(j+1)} (1-t)^j. \tag{10} $$

Identities (7) and (9) could be found in [11]. Identities (8) and (10) could be derived using the properties of ${}_t I_1^\alpha$, ${}_t D_1^\alpha x(t)$ and ${}_t^C D_1^\alpha x(t)$. On the other hand, one can also prove (8) and (10) by applying the reflection operator Q on (7) and (9) [9].

Identities (7)–(10) will play a crucial role in analytical schemes discussed here for Type2 FDEs for which we also need terminal conditions. In FVC, the Euler–Lagrange equations and transversality conditions provide the necessary FDEs and natural boundary conditions, respectively. For this reason, we briefly provide a summary of the Euler–Lagrange equation and transversality conditions for a simple FVP. Further details could be found in [13].

3. The Euler–Lagrange equation and the transversality conditions

In this section, we briefly review the Euler–Lagrange equations and transversality conditions for a simple FVP. For simplicity in the discussion to follow, we assume from here onwards that $0 < \alpha < 1$. The Euler–Lagrange equations and transversality conditions for an arbitrary α greater than zero could be found in [13].

A simple FVP can be described as follows: among all possible functions $x(t)$, find the function $x^*(t)$ which minimizes the functional

$$ J[x] = \int_0^1 F\left(t, x, {}_0^C D_1^\alpha x\right) \mathrm{d}t. \tag{11} $$

This problem leads to the following Euler–Lagrange equation [13]:

$$ \frac{\partial F}{\partial x} + {}_t D_1^\alpha \frac{\partial F}{\partial {}_0^C D_t^\alpha x} = 0 \tag{12} $$

and the following transversality condition

$$ \left({}_t D_1^{\alpha-1} \frac{\partial F}{\partial {}_0^C D_t^\alpha x} \right) \delta x(t) \Big|_0^1 = 0. \tag{13} $$

Note that the functional contains only a left Caputo fractional derivative, whereas the associated Euler–Lagrange equation contains a right RL derivative also. If the functional contained a right Caputo fractional derivative, then the associated Euler–Lagrange equation would also contain a left RL derivative. Similarly, if the functional were defined in terms of a left and a right RL derivatives, then the resulting Euler–Lagrange equation would have contained left and right Caputo derivatives. The point to be made here is that an FDE resulting from an FVP would contain both the left and the right fractional derivatives. This establishes the need for developing analytical schemes for Type2 FDEs. Also note that according to (13) at $t = 0$ either $x(t)$ should be specified or we must have

$$\left({}_tD_1^{\alpha-1} \frac{\partial F}{\partial {}_0^CD_t^\alpha x} \right)\Big|_0 = 0. \tag{14}$$

The same applies at point $t = 1$. Thus, (13) provides a clue as to what type of boundary conditions should be considered to solve an FDE.

We now present analytical solutions for a class of Type2 FDEs.

4. Analytical solutions for a class of Type2 FDEs

Our aim is to solve Type2 FDEs of the following kind:

$${}_tD_1^\alpha\left(a_1(t){}_0^CD_t^\alpha x\right) + {}_tD_1^\alpha(a_2(t)x) + {}_0^CD_t^\alpha(a_3(t)x) + a_4(t)x + {}_tD_1^\alpha a_5(t) + a_6(t) = 0, \tag{15}$$

where $a_1(t), a_2(t), a_3(t), a_4(t), a_5(t)$ and $a_6(t)$ are functions of time. In a more general setting, they would also contain function $x(t)$. This class of FDEs may come from FVC or from some other physical principles. To the author's knowledge, analytical schemes for such a general Type2 FDEs have not been presented. Here, we will present analytical schemes for some specialized class of Type2 FDEs. From here onward, it will be implicitly assumed that FDEs considered are of Type2.

We begin with a simple FDE.

4.1. Analytical solution of a simple FDE

As the first example, consider the following FDE:

$${}_tD_1^\alpha\left({}_0^CD_t^\alpha x(t)\right) = 1. \tag{16}$$

In this case, $a_1(t) = -a_6(t) = 1$ and $a_2(t) = a_3(t) = a_4(t) = a_5(t) = 0$. Equation (16) is perhaps the simplest FDE, because in this case the mixed derivative term is a constant. It can be obtained by taking the Lagrangian $F\left(t, x, {}_0^CD_t^\alpha x\right)$ as

$$F\left(t, x, {}_0^CD_t^\alpha x\right) = \tfrac{1}{2}\left[\left({}_0^CD_t^\alpha x\right)^2 - x\right] \tag{17}$$

and using (12). This problem was considered in [13], where the FDE (16) was derived using (12) and (17), and the problem was solved using a numerical technique. Here, we present an analytical technique for this problem.

To solve this problem, we consider the following initial condition:

$$x(0) = e_0, \tag{18}$$

where e_0 is a constant. Equation (13) then requires that

$${}_tD_1^{\alpha-1}\left({}_0^CD_t^\alpha x(t)\right)\big|_{x=1} = 0. \tag{19}$$

In the discussion to follow, we assume that

$${}_tD_1^{\alpha-1}\left({}_0^CD_t^\alpha x(t)\right)\big|_{x=1} = e_1, \tag{20}$$

where e_1 is a constant that need not be 0; the rationale for which will be given in the next subsection. Note that this does not preclude us from taking $e_1 = 0$. Applying ${}_tI_1^\alpha$ on both sides of (16), and using (8) and (20), we obtain

$$ {}_0^CD_t^\alpha x(t) = \frac{(1-t)^\alpha}{\Gamma(1+\alpha)} + \frac{e_1}{\Gamma(\alpha)}(1-t)^{\alpha-1}. \tag{21} $$

Applying ${}_0I_t^\alpha$ on both sides of (21), and using (9) and (20), we obtain

$$ x(t) = e_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\frac{(1-\tau)^\alpha}{\Gamma(1+\alpha)} + \frac{e_1}{\Gamma(\alpha)}(1-\tau)^{\alpha-1}\right]\mathrm{d}\tau. \tag{22} $$

Equation (22) provides solution of (16).

We now consider analytical solution of a simple fractional oscillator.

4.2. *Analytical solution of a simple fractional oscillator*

We now consider the following FDE of a fractional oscillator:

$$ {}_tD_1^\alpha\left({}_0^CD_t^\alpha x(t)\right) = \omega_\alpha^2 x(t), \tag{23} $$

where ω_α is the frequency of the oscillator. In this case, $a_1(t) = 1, a_4(t) = -\omega_\alpha^2$ and $a_2(t) = a_3(t) = a_5(t) = a_6(t) = 0$. Equation (23) can be obtained by taking the Lagrangian $F\left(t, x, {}_0^CD_t^\alpha x\right)$ as

$$ F\left(t, x, {}_0^CD_t^\alpha x\right) = \tfrac{1}{2}\left[\left({}_0^CD_t^\alpha x\right)^2 - \omega_\alpha^2 x^2\right] \tag{24} $$

and using (12). For $\alpha = 1$, (23) and (24) represent the differential equation of motion and a Lagrangian of a one-dimensional oscillator and $\omega_1 = \omega$ its frequency. For this reason, for an arbitrary α (23) and (24) are called an FDE and a Lagrangian of a fractional oscillator.

Solutions of FDEs of fractional oscillators have been considered by Narahari Achar and coworker, Tofighi and others (see [18, 19] and the references therein). However, in those papers, the differential equations of the fractional oscillators are obtained by replacing the second derivative term with a forward fractional derivative of order 2α. In contrast, our equation for the oscillator comes from a fractional Lagrangian and it contains both the left and the right derivatives.

As in section 4.1, we assume that the following initial condition:

$$ x(0) = e_0, \tag{25} $$

where e_0 is a constant. Equation (13) then requires that

$$ {}_tD_1^{\alpha-1}\left({}_0^CD_t^\alpha x(t)\right)\Big|_{x=1} = 0. \tag{26} $$

We assume that

$$ {}_tD_1^{\alpha-1}\left({}_0^CD_t^\alpha x(t)\right)\Big|_{x=1} = e_1, \tag{27} $$

where e_1 is a constant which need not be 0. This assumption requires an explanation. Note that for $\alpha = 1$, the natural boundary condition requires that $\dot{x}(1)$ be 0. However, in dynamics, $\dot{x}(0)$ is considered specified, and it need not be 0. Although the resulting trajectory provides a solution of the differential equation for the oscillator, it does not give minimum of the functional. The assumption in (28) is made in that spirit. This also provides a justification for considering e_1 an arbitrary real number in (20). Of course, if $e_1 \neq 0$, the resulting trajectory will not give the minimum of the functional.

To find the solution of the fractional oscillator, we define

$$ {}_0^CD_t^\alpha x(t) = \omega_\alpha y(t). \tag{28} $$

Substituting (28) into (23), we obtain

$$ {}_tD_1^\alpha y(t) = \omega_\alpha x(t). \tag{29}$$

We now apply ${}_tI_1^\alpha$ on both sides of (29), and use (8) and (27), to obtain

$$\omega_\alpha y(t) = {}_0^CD_t^\alpha x(t) = \omega_\alpha^2 {}_tI_1^\alpha x(t) + \frac{e_1\omega_\alpha}{\Gamma(\alpha)}(1-t)^{\alpha-1}. \tag{30}$$

Similarly, we apply ${}_0I_t^\alpha$ on both sides of (28), and use (9), (25) and (29) to obtain [9]

$$x(t) = \omega_\alpha^2 {}_0I_t^\alpha {}_tI_1^\alpha x(t) + \frac{e_1\omega_\alpha}{\Gamma(\alpha)} {}_0I_t^\alpha (1-t)^{\alpha-1} + c_0. \tag{31}$$

Equation (31) can be thought of as a Volterra-type integral equation that has composite integral operators. This equation can be solved by using successive approximation or Heaviside operational approach [6, 11]. Following the operational approach, we obtain

$$\begin{aligned} x(t) &= \left(1 - \omega_\alpha^2 {}_0I_t^\alpha {}_tI_1^\alpha\right)^{-1} \left(c_0 + \frac{e_1\omega_\alpha}{\Gamma(\alpha)} {}_0I_t^\alpha (1-t)^{\alpha-1}\right) \\ &= \sum_{j=0}^{\infty} \left(\omega_\alpha^2 {}_0I_t^\alpha {}_tI_1^\alpha\right)^j \left(c_0 + \frac{e_1\omega_\alpha}{\Gamma(\alpha)} {}_0I_t^\alpha (1-t)^{\alpha-1}\right). \end{aligned} \tag{32}$$

Equation (32) provides solution of the FDE defined by (23) and the terminal conditions (25) and (27). Note that the terminal conditions are linearly related to other terminal conditions. Therefore, the above solutions can still be used to find solutions for other type of terminal conditions.

4.3. Analytical solution of a general class of FDEs

As the third example, we consider the following FDE:

$$ {}_tD_1^\alpha \left(a_1 {}_0^CD_t^\alpha x + a_3 x\right) + a_3 {}_0^CD_t^\alpha x + a_4 x + {}_tD_1^\alpha a_5(t) + a_6(t) = 0. \tag{33}$$

This equation is obtained from (15) by taking $a_1(t)$, $a_2(t)$, $a_3(t)$ and $a_4(t)$ as constants, and replacing a_2 by a_3. Functions $a_5(t)$ and $a_6(t)$ can still be functions of time. It can be demonstrated that (33) can be derived by taking the Lagrangian $F\left(t, x, {}_0^CD_t^\alpha x\right)$ as

$$F\left(t, x, {}_0^CD_t^\alpha x\right) = \frac{a_1}{2}\left({}_0^CD_t^\alpha x\right)^2 + \frac{a_4}{2}x^2 + a_3 x {}_0^CD_t^\alpha x + a_5(t) {}_0^CD_t^\alpha x + a_6(t) x \tag{34}$$

and using (12). We further restrict ourselves to the following conditions $a_4 = \lambda_1 a_3$ and $a_3 = \lambda_1 a_1$. Thus, our formulation presented here is limited in scope. A general condition, where a_1, a_2, a_3 and a_4 are functions of time with no additional restrictions on them will be attempted in the future. With these assumptions, (33) reduces to

$$ {}_tD_1^\alpha y(t) + \lambda_1 y(t) = f(t), \tag{35}$$

where

$$ {}_0^CD_t^\alpha x + \lambda_2 x = \frac{1}{a_1} y(t) \tag{36}$$

and

$$ f(t) = -\left({}_tD_1^\alpha a_5(t) + a_6(t)\right) \tag{37}$$

with

$$\lambda_2 = \frac{a_3}{a_1}. \tag{38}$$

As before, we assume the following initial condition:

$$x(0) = e_0, \tag{39}$$

where e_0 is a constant. Equation (13) then requires that

$$ {}_tD_1^{\alpha-1}\big(a_1{}_0^cD_t^\alpha x(t) + a_3x + a_5(t)\big)\big|_{x=1} = {}_tD_1^{\alpha-1}(y + a_5(t))\big|_{x=1} = 0. \tag{40}$$

Thus, we assume that

$$ {}_tD_1^{\alpha-1}y\big|_{x=1} = e_1 \tag{41}$$

where e_1 is a constant which need not be equal to $-{}_tD_1^{\alpha-1}a_5(t)\big|_{x=1}$. Of course, this does not preclude us from taking $e_1 = -{}_tD_1^{\alpha-1}a_5(t)\big|_{x=1}$. If this condition is not satisfied, then the resulting solution will not provide a minimum. Applying ${}_tI_1^\alpha$ on both sides of (35), and using (8) and (41), we obtain

$$y(t) + \lambda_1{}_tI_1^\alpha y(t) = \frac{e_1}{\Gamma(\alpha)}(1-t)^{\alpha-1} + {}_tI_1^\alpha f(t). \tag{42}$$

Equation (42) is a Volterra-type integral equation that can be solved using successive approximation or using Heaviside operational approach [11]. Using operational approach and a semigroup property of the operators ${}_tI_1^{j\alpha}$, $j = 0, \ldots, \infty$, we obtain

$$y(t) = e_1(1-t)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_1(1-t)^\alpha] + \int_t^1 (\tau - t)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_1(\tau-t)^\alpha]f(\tau)\,\mathrm{d}\tau, \tag{43}$$

where

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j + \beta)} \tag{44}$$

is the generalized Mittag–Leffler function. Applying ${}_0I_t^\alpha$ on both sides of (36), and using (9) and (39), we obtain

$$x(t) + \lambda_2{}_0I_t^\alpha x(t) = c_0 + \frac{1}{a_1}{}_0I_t^\alpha y(t). \tag{45}$$

Equation (45) is also a Volterra-type integral equation.

Using operational approach and a semigroup property of the operators ${}_0I_t^{j\alpha}$, $j = 0, \ldots, \infty$, we obtain

$$x(t) = c_0E_{\alpha,1}[-\lambda_2t^\alpha] + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_2(t-\tau)^\alpha]y(\tau)\,\mathrm{d}\tau. \tag{46}$$

Substituting $y(t)$ from (43) into (46), we get the solution for $x(t)$.

As a special case, consider that $a_5(t) = (1-t)^{\beta-1}$ and $a_6(t) = (1-t)^{\gamma-\alpha-1}$. In this case, using (37), $f(t)$ is given as

$$f(t) = -\left[\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(1-t)^{\beta-\alpha-1} + (1-t)^{\gamma-\alpha-1}\right]. \tag{47}$$

Using (47), (42) reduces to

$$\begin{aligned} y(t) = {} & e_1(1-t)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_1(1-t)^\alpha] - [\Gamma(\beta)(1-t)^{\beta-1}E_{\alpha,\beta}[-\lambda_1(1-t)^\alpha] \\ & + \Gamma(\gamma-\alpha)(1-t)^{\gamma-1}E_{\alpha,\gamma}[-\lambda_1(1-t)^\alpha]]. \end{aligned} \tag{48}$$

Here, $y(t)$ has been written explicitly in terms of t. Moreover, it can be written explicitly in terms of power series. Substituting this into (46), we can obtain $x(t)$ in terms of hypergeometric functions.

We now consider an FDE for which a Lagrangian may not exist.

4.4. Analytical solution of an FDE not amenable to a Lagrangian

In this section, we consider an FDE for which a Lagrangian may not exist. Specifically, we consider the FDEs of the following type:

$$ {}_tD_1^{\alpha}\left({}_0^C D_t^{\alpha} x(t)\right) + a_2(t){}_0^C D_t^{\alpha} x(t) = f(t). \quad (49) $$

Our effort suggests that this FDE cannot be obtained using a Lagrangian and the Euler–Lagrange equation. We consider the following terminal conditions:

$$ x(0) = e_0 \quad (50) $$

and

$$ {}_tD_1^{\alpha-1}\left({}_0^c D_t^{\alpha} x(t)\right)\Big|_{x=1} = e_1. \quad (51) $$

We define

$$ {}_0^C D_t^{\alpha} x(t) = y(t) \quad (52) $$

and rewrite equation (49) as

$$ {}_tD_1^{\alpha} y(t) + a_2(t)y(t) = f(t). \quad (53) $$

Applying ${}_tI_1^{\alpha}$ from right on both sides of equation (53), we obtain

$$ y(t) + {}_tI_1^{\alpha}(a_2(t)y(t)) = \frac{e_1}{\Gamma(\alpha)}(1-t)^{\alpha-1} + {}_tI_1^{\alpha} f(t). \quad (54) $$

Equation (54) is also a Volterra-type integral equation that can be solved using successive approximation or using operational approach. Applying ${}_0I_t^{\alpha}$ on both sides of equation (52), we obtain

$$ x(t) = e_0 + {}_0I_t^{\alpha} y(t), \quad (55) $$

where $y(t)$ is given by equation (54).

We now consider two special cases.

Case 1: $a_2(t) = \lambda$ and arbitrary $f(t)$.

In this case, equation (54) reduces to

$$ y(t) = e_1(1-t)^{\alpha-1}E_{\alpha,\alpha}[-\lambda(1-t)^{\alpha}] + \int_t^1 (\tau - t)^{\alpha-1}E_{\alpha,\alpha}[-\lambda(1-t)^{\alpha}]f(\tau)\,\mathrm{d}\tau. \quad (56) $$

Equation (56) can be obtained using a semigroup property of the operators ${}_tI_1^{j\alpha}$, $j = 0, \ldots, \infty$, or the method of successive approximation [11].

Case 2: $a_2(t) = -\lambda(1-t)^{\beta}$ and $f(t) = 0$.

In this case, equation (54) reduces to

$$ y(t) = \frac{e_1}{\Gamma(\alpha)}(1-t)^{\alpha-1}E_{\alpha,(\alpha+\beta)/\alpha,(\alpha+\beta-1)/\alpha}[-\lambda(1-t)^{\alpha+\beta}], \quad (57) $$

where $E_{\alpha,\beta,\gamma}(z)$ is a function defined as [11]

$$ E_{\alpha,\beta,\gamma}(z) = \sum_{k=0}^{\infty} c_k z^k \quad (58) $$

with

$$ c_0 = 1, \qquad \text{and} \qquad c_k = \Pi_{j=0}^{k-1}\frac{\Gamma[\alpha(j\beta+\gamma)+1]}{\Gamma[\alpha(j\beta+\gamma+1)+1]}. \quad (59) $$

In both cases, $x(t)$ is obtained by substituting $y(t)$ in (55).

5. Conclusions

Fractional differential equations that come from fractional variational calculus contain both the left and the right fractional derivatives. These FDEs may also come from other physical principles. Analytical schemes were presented to solve these types of FDEs. The methods utilized properties of the fractional integrals, operational approach and successive approximation technique to obtain the solutions. It is hoped that this investigation will initiate further research in this field, and more elegant schemes would be found to solve Type2 FDEs.

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